

Strings in Twistor Superspace and Mirror Symmetry

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We obtain the super-Landau-Ginzburg mirror of the A-twisted topological sigma model on a twistor superspace – the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ which is a Calabi-Yau supermanifold. We show that the B-model mirror has a geometric interpretation. In a particular limit for one of the Kähler parameters of the quadric, we show that the mirror can be interpreted as the twistor superspace $\mathbb{CP}^{3|4}$. This agrees with the recent conjecture of Neitzke and Vafa proposing a mirror equivalence between the two twistor superspaces.

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1. Introduction

Recently Witten [1] has argued that perturbative $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory can be formulated as topological string theory with the supertwistor space $\mathbb{CP}^{3|4}$ as target. Among other things this has led to new and interesting observations about certain Calabi-Yau supermanifolds [2][3] which happen to be supertwistor spaces and these form the focus of our note.

In particular, in [1] it was demonstrated that the $\mathcal{N} = 4$ Yang-Mills amplitudes, when transformed to the supertwistor space $\mathbb{CP}^{3|4}$, are supported on holomorphic curves which were then interpreted as D1-instantons of the topological B-model on $\mathbb{CP}^{3|4}$. On the other hand it was shown long ago [4] that the classical equations of motion of the $\mathcal{N} = 4$ gauge theory follow from integrability of gauge fields on supersymmetric lightlike lines. The space of all such lightlike lines in (complexified) compactified Minkowski space is the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. It is natural to ask if there is some relation between these two pictures. In [2] it was conjectured that these two twistorial formulations could possibly be related by mirror symmetry between $\mathbb{CP}^{3|4}$ and the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. (They are both Calabi-Yau supermanifolds). This conjecture was prompted by a combination of two observations. First, the authors of [2] argued that the $\mathcal{N} = 4$ Yang-Mills amplitudes could also be obtained from the A-model topological string on $\mathbb{CP}^{3|4}$ to be understood as S -dual (see also [5] and [6]) to the B-model picture of [1]. In this picture the D1-instantons of [1] are replaced by worldsheet instantons of the A-model (and the D5-branes by NS5-branes). Secondly, given such an A-model description, it is natural to expect that a potential mirror B-model description will have no instantons and the perturbative Yang-Mills amplitudes will be realized classically. The observations of [4] outlined above suggest a candidate for such a mirror. Specifically, one expects that the B(A)-model on $\mathbb{CP}^{3|4}$ is mapped by a mirror transformation to the A(B)-model on the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. In a recent work [3] it was demonstrated that the A-model on the former supermanifold is mirror to the B-model on the latter in the limit where the Kähler class t of $\mathbb{CP}^{3|4}$ is sent to minus infinity.

In this note we show that the A-model on the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ is mirror to the B-model on $\mathbb{CP}^{3|4}$ in a certain limit for one of the two Kähler parameters of the quadric. The motivation for this study is two-fold. One would like to understand if indeed the two twistorial formulations above are related by mirror symmetry, independently of the conjectured S -duality for topological strings. Further, one would like to shed light

on the relationship between Kähler class and complex structure deformations of the two supermanifolds in question. It should be pointed out that we are studying the closed topological model (without any extra branes) which corresponds to the gravitational theory. Our results show that the B-model mirror of the quadric should be understood as a complex deformation of the twistor superspace $\mathbb{CP}^{3|4}$ (with a line at infinity removed). It has been argued that complex deformations of twistor space get mapped to \mathbb{C}^4 (complexified Minkowski space) with points blown up (see [5] and references therein). It would be extremely interesting to develop this idea further.

In the following section we review some essential results in the context of mirror symmetry for supermanifolds. In Section 3 we apply these to the A-model on the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ and obtain the mirror B-model which has a geometric interpretation.

2. Supermanifolds and Hypersurfaces in Toric Manifolds

We begin by reviewing the results of [7] and [3] which are relevant for our computation. We are interested in computing the mirror transform of the topological sigma model of the A-type on the quadric ³ which is realized as a hypersurface in a toric supermanifold.

2.1. Degree d hypersurface in \mathbb{CP}^{d-1}

To understand how this proceeds we first recall the well-known fact [8] that the observables of the A-model on a bosonic Calabi-Yau manifold \mathcal{M} realized as a hypersurface in a compact toric manifold are simply related to the observables of the A-model on a corresponding *non-compact* toric manifold V . As a simple example, consider a hypersurface obtained from a degree d polynomial equation in \mathbb{CP}^{d-1} . This is realised as a $U(1)$ -gauged linear sigma model with $\mathcal{N} = (2, 2)$ supersymmetry and d chiral superfields $\{\Phi_i\}$ of charge $+1$ each. In addition there is a field P of charge $-d$ and a superpotential,

$$W = \epsilon P G(\Phi_i) \tag{2.1}$$

where $G(\Phi_i)$ is a degree d polynomial (weight d) [9]. These lead in the infrared to a non-linear sigma model description with the Calabi-Yau \mathcal{M} as target via the vacuum equations

$$G(\Phi_i) = 0; \quad P = 0, \tag{2.2}$$

³ For the sake of brevity we will refer to the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ simply as “the quadric”.

modulo complex gauge transformations. However, the observables of the A-model (the (a,c) -ring) cannot depend on the superpotential which is a (c,c) -ring deformation. Therefore we expect that as $\epsilon \rightarrow 0$ and the superpotential disappears, the observables of the A-model on \mathcal{M} are simply related to the observables of the *non-compact* Calabi-Yau manifold V which is the $\mathcal{O}(-d)$ line bundle over \mathbb{CP}^{d-1} (the field P is then identified with the coordinate on the fibre). Of course, this reasoning is not strictly correct since the two theories differ drastically, as the one with vanishing superpotential has 2 extra complex dimensions. Nevertheless, one indeed obtains the following correspondence between states of the A-twisted theory on \mathcal{M} and those on V ,

$$|\Sigma\rangle_V \rightarrow |\mathbf{1}\rangle_{\mathcal{M}} \quad (2.3)$$

where Σ is the twisted chiral abelian field strength and the state $|\Sigma\rangle_V$ is a normalizable ground state of the noncompact theory obtained by an insertion of Σ which corresponds to the Kähler form controlling the size of the compact part of the geometry. Alternatively, such Σ -insertions can be achieved by taking derivatives of correlators of the A-twisted model on V with respect to the complexified Kähler parameter t , yielding A-model observables on the compact manifold \mathcal{M} .

2.2. Realization as the supermanifold $\mathbb{WCP}^{d-1|1}(1, \dots, 1|d)$

It was shown in [7] that the above procedure of computing A-model observables of the compact manifold \mathcal{M} is equivalent to A-model computations on a $(d-1|1)$ -dimensional toric *supermanifold*. Specifically, in the example above, it simply amounts to replacing P by a fermionic chiral superfield Ψ of charge d . Thus we have an $\mathcal{O}(d)$ (fermionic) bundle over \mathbb{CP}^{d-1} which can also be viewed as the toric supermanifold $\hat{\mathcal{M}}$ namely, $\mathbb{WCP}^{d-1|1}(1, \dots, 1|d)$. The D-term constraint is now,

$$\sum_{i=1}^d |\Phi_i|^2 + d \bar{\Psi} \Psi = \text{Re}[t] \quad (2.4)$$

where the Kähler class parameter of $\hat{\mathcal{M}}$ is the same as that of the compact Calabi-Yau \mathcal{M} . Since $\hat{\mathcal{M}}$ has $U(1)$ isometries one can perform T-duality to obtain the super-Landau-Ginzburg mirror. Techniques for doing this have been discussed in [10] and [3]. Interestingly the upshot of this procedure is that the super-Landau-Ginzburg directly yields the observables, such as periods of the compact bosonic Calabi-Yau \mathcal{M} . Put another way, the fermionic fields of the sigma model on $\hat{\mathcal{M}}$ automatically incorporate the projection (the t -derivative) that was required above to translate the observables of the non-compact manifold V into those of the compact Calabi-Yau \mathcal{M} .

2.3. Review of T-duality for fermionic coordinates

The implementation of T-duality for fermionic coordinates has only recently been discussed in [3]. Since it is not part of standard literature we review the main results here which will be used subsequently. Just as in the case of bosonic coordinates with a $U(1)$ isometry [11] we wish to dualize the phase for a fermionic superfield Ψ with a $U(1)$ -charge q . The phase is bosonic and hence it will dualize into a bosonic twisted chiral multiplet Y . The real part of Y is determined as $Y + \bar{Y} = \bar{\Psi}\Psi$ while its imaginary part is periodic. In addition, the usual twisted chiral superpotential is also generated for Y which gives the winding modes a mass $q\Sigma$. However, this is not all. The original theory had one fermionic coordinate Ψ whose momentum modes have mass $q\Sigma$. Hence the dualized theory cannot simply have one bosonic degree of freedom. In fact, it should have two fermionic superfields η, χ with the same mass $q\Sigma$ as the winding modes of the dual bosonic coordinate Y . This ensures that one boson and one fermion cancel in the partition function. In sum then, the T-dual of the fermionic superfield Ψ yields the bosonic twisted chiral multiplet Y and two fermion superfields η, χ with a superpotential

$$W = -q \Sigma(Y - \eta\chi) + e^{-Y} \quad (2.5)$$

This superpotential gives the same mass $-d \Sigma$ to the winding modes of Y and the excitations of η, χ . We can rewrite this superpotential in a different form after a shift $Y \rightarrow Y + \eta\chi$ so that

$$W = -q \Sigma Y + e^{-Y}(1 - \eta\chi). \quad (2.6)$$

Now we see what the effect of the fermions is on the partition function of the dual theory. Integrating them out brings down a factor of e^{-Y} in the measure turning e^{-Y} into a good coordinate. In the context of the example discussed above this is precisely the effect of taking a t -derivative of the partition function of the theory corresponding to the bosonic non-compact manifold V .

3. Mirror of the quadric

We are now ready to apply the results above to the case of interest. We take two copies of $\mathbb{CP}^{3|3}$ each with the homogeneous coordinates $\{X_I, \Psi_A\}$ and $\{\tilde{X}_I, \tilde{\Psi}_A\}$ respectively where $\{X_I, \tilde{X}_I\}$, ($I = 1, \dots, 4$) are bosonic coordinates and $\{\Psi_A, \tilde{\Psi}_A\}$, ($A = 1, 2, 3$) are

the fermionic coordinates. Let t_1 and t_2 be the complexified Kähler class parameters for the two $\mathbb{CP}^{3|3}$'s. The quadric Q in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ is then defined by the bilinear equation

$$G := \sum_{I=1}^4 X_I \tilde{X}_I + \sum_{A=1}^3 \Psi_A \tilde{\Psi}_A = 0 \quad (3.1)$$

There is a \mathbf{Z}_2 symmetry which exchanges the two $\mathbb{CP}^{3|3}$ s and their Kähler classes $t_1 \leftrightarrow t_2$. This quadric can be realised as a $U(1) \times U(1)$ gauged linear sigma model with the charge assignments $(1, 0)$ for the fields $\{X_I, \Psi_A\}$ and $(0, 1)$ for the second copy of coordinates $\{\tilde{X}_I, \tilde{\Psi}_A\}$. In addition we introduce a bosonic chiral superfield P of charge $(-1, -1)$ and a superpotential

$$W = \epsilon P G[X_I, \tilde{X}_I, \Psi_A, \tilde{\Psi}_A]. \quad (3.2)$$

The field content and charge assignments ensure that there is no $U(1)_A$ anomaly and the Calabi-Yau condition is satisfied. (Note that due to the reversed statistics the fermionic coordinates contribute to the anomaly with a sign opposite to that of the bosonic coordinates.) The theory flows to a (super)Calabi-Yau phase where it is a non-linear sigma model with the quadric Q as target, corresponding to the vacuum manifold

$$G[X_I, \tilde{X}_I, \Psi_A, \tilde{\Psi}_A] = 0 \quad (3.3)$$

with $P = 0$ in the D-term constraints,

$$\begin{aligned} \sum_{I=1}^4 |X_I|^2 + \sum_{A=1}^3 |\Psi_A|^2 - |P|^2 &= \text{Re}[t_1] \\ \sum_{I=1}^4 |\tilde{X}_I|^2 + \sum_{A=1}^3 |\tilde{\Psi}_A|^2 - |P|^2 &= \text{Re}[t_2] \end{aligned} \quad (3.4)$$

modulo $U(1) \times U(1)$ gauge transformations ($\text{Re}[t_1], \text{Re}[t_2] > 0$). The quadric is a hypersurface in a $(6|6)$ -dimensional toric supermanifold. The bosonic hypersurface equation (3.3) means that the quadric Q has complex (super)dimension $(5|6)$.

In order to implement the mirror transform we need to realise the topological A-model observables on Q in terms of a sigma model on a toric (super)manifold *i.e.* one without the superpotential which imposes the hypersurface constraint. One is naturally tempted to use the ideas of [8] outlined in the previous section namely, to study the A-model on the “non-compact” Calabi-Yau where we send $W \rightarrow 0$. However, this immediately leads to a puzzle – the sigma model with $W = 0$ has a $(7|6)$ -dimensional target space. On the

other hand the quadric has bosonic minus fermionic dimension -1 and its mirror must naturally have dimension $(n-1|n)$. The resolution is straightforward and we simply need to employ the ideas of [7] and [3] as explained earlier. We must not only send W to zero but we must also replace the bosonic field P with a fermionic chiral superfield Ψ_P with charge $(1, 1)$ under the $U(1) \times U(1)$ gauge symmetry.

In summary, the topological A-model on the quadric Q is equivalent to the A-model on the $\mathcal{O}_1(1) \otimes \mathcal{O}_2(1)$ fermionic bundle over $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ (by $\mathcal{O}_1(1)$ we mean the pullback of the $\mathcal{O}(1)$ line bundle on the first $\mathbb{CP}^{3|3}$ factor, and similarly for the second).

3.1. B-model mirror

The Landau-Ginzburg B-model dual of the above can be obtained as follows. T-duality replaces each bosonic superfield X_I and \tilde{X}_I with the cylinder-valued coordinates Y_I and \tilde{Y}_I respectively, ($I = 1, \dots, 4$). Further, using the rules for dualizing the fermionic coordinates where each such field yields a bosonic coordinate and a pair of fermionic fields, the $\{\Psi_A\}$ and $\{\tilde{\Psi}_A\}$ dualize to the set $\{M_A, \eta_A, \chi_A\}$ and $\{\tilde{M}_A, \tilde{\eta}_A, \tilde{\chi}_A\}$ respectively, ($A = 1, 2, 3$). In our notation the η 's and χ 's are fermion superfields. Finally, the A-model fermion Ψ_P with charge $(1, 1)$ dualizes to (Y_P, η, χ) . The Landau-Ginzburg mirror of the quadric is given by the path integral (for the holomorphic sector)

$$\begin{aligned}
Z = & \int \prod_{I=1}^4 [dY_I d\tilde{Y}_I] \prod_{A=1}^3 [dM_A d\tilde{M}_A d\eta_A d\chi_A d\tilde{\eta}_A d\tilde{\chi}_A] dY_P d\eta d\chi \\
& \delta(\sum Y_I - \sum M_A - Y_P - t_1) \delta(\sum \tilde{Y}_I - \sum \tilde{M}_A - Y_P - t_2) \\
& \times \exp \left[\sum e^{-Y_I} + \sum e^{-\tilde{Y}_I} + \sum e^{-M_A} (1 + \eta_A \chi_A) + \sum e^{-\tilde{M}_A} (1 + \tilde{\eta}_A \tilde{\chi}_A) + e^{-Y_P} (1 + \eta \chi) \right]
\end{aligned}
\tag{3.5}$$

The Landau-Ginzburg model has 13 bosonic (taking into account the two delta-function constraints) and 14 fermionic degrees of freedom. Note that the \mathbf{Z}_2 exchange symmetry of the quadric is explicit in the B-model partition function above. To arrive at a mirror super-Calabi-Yau interpretation for this Landau-Ginzburg we perform a sequence of manipulations that involve integrating out some of the fields and successive field redefinitions.

We first integrate out the fermions $\tilde{\eta}_A, \tilde{\chi}_A, \eta_3, \chi_3, \eta, \chi$, and solve the delta-function constraints for Y_P and M_3 . This breaks the symmetry that exchanges the two $\mathbb{CP}^{3|3}$'s of

the A-model. We will come back to this point later. At this stage we have the following B-model integral with 4 fermions and 13 bosons

$$\begin{aligned}
Z = & \int \prod_{I=1}^4 [dY_I d\tilde{Y}_I] dM_1 dM_2 \prod_{A=1}^3 [d\tilde{M}_A] d\eta_1 d\chi_1 d\eta_2 d\chi_2 e^{M_1+M_2-\sum Y_I-\sum \tilde{M}_A+t_1} \\
& \times \exp \left[\sum e^{-Y_I} + \sum e^{-\tilde{Y}_I} + e^{-M_1}(1+\eta_1\chi_1) + e^{-M_2}(1+\eta_2\chi_2) + \sum e^{-\tilde{M}_A} + \right. \\
& \left. + e^{t_1-t_2} e^{M_1+M_2-\sum Y_I+\sum \tilde{Y}_I-\sum \tilde{M}_A} + e^{t_2} e^{\sum \tilde{M}_A-\sum \tilde{Y}_I} \right].
\end{aligned} \tag{3.6}$$

We see that integrating out the fermions leads to non-trivial factors in the measure. These measure factors turn the fields e^{-Y_I} and $e^{-\tilde{M}_A}$ which were \mathbb{C}^* -valued, into good coordinates, so that we can define the new \mathbb{C} -valued fields $y_I = e^{-Y_I}$ and $\tilde{m}_A = e^{-\tilde{M}_A}$. In fact it is convenient to make a similar change of variables for all the bosonic fields:

$$m_{1,2} := e^{-M_{1,2}}; \quad \tilde{m}_A := e^{-\tilde{M}_A}; \quad y_I := e^{-Y_I}; \quad \tilde{y}_I := e^{-\tilde{Y}_I}. \tag{3.7}$$

In terms of these new fields the Landau-Ginzburg model is

$$\begin{aligned}
Z = & e^{t_1} \int \prod_{I=1}^4 [dy_I] \prod_{I=1}^4 \left[\frac{d\tilde{y}_I}{\tilde{y}_I} \right] \frac{dm_1 dm_2}{m_1^2 m_2^2} \prod_{A=1}^3 [d\tilde{m}_A] d\eta_1 d\chi_1 d\eta_2 d\chi_2 \\
& \exp \left[\sum_{I=1}^4 y_I + \sum_{I=1}^4 \tilde{y}_I + \sum_{a=1}^2 m_a (1 + \eta_a \chi_a) + \sum_{A=1}^3 \tilde{m}_A + e^{t_2} \frac{\prod_I \tilde{y}_I}{\prod_A \tilde{m}_A} + e^{t_1-t_2} \frac{\prod_A \tilde{m}_A \prod_I y_I}{m_1 m_2 \prod_I \tilde{y}_I} \right].
\end{aligned} \tag{3.8}$$

A further change of variable

$$\tilde{x}_A := \frac{\tilde{y}_A}{\tilde{m}_A}, \quad (A = 1, 2, 3); \quad \tilde{x}_4 := \tilde{y}_4; \quad x_a := \frac{y_a \tilde{m}_a}{\tilde{y}_a m_a}, \quad (a = 1, 2); \quad x_b := \frac{y_b}{\tilde{x}_b} \quad (b = 3, 4) \tag{3.9}$$

allows us to bring the Landau-Ginzburg superpotential in the exponent in (3.8) to a polynomial form which will lead us to the interpretation as a super-Calabi-Yau manifold:

$$\begin{aligned}
Z = & e^{t_1} \int \prod_{I=1}^4 [dx_I] \prod_{I=1}^4 [d\tilde{x}_I] \frac{dm_1}{m_1} \frac{dm_2}{m_2} \prod_{A=1}^3 [d\tilde{m}_A] d\eta_1 d\chi_1 d\eta_2 d\chi_2 \\
& \exp \left[\sum_{A=1}^3 \tilde{m}_A (1 + \tilde{x}_A) + \sum_{a=1,2} m_a (1 + x_a \tilde{x}_a + \eta_a \chi_a) + x_3 \tilde{x}_3 + x_4 \tilde{x}_4 + \right. \\
& \left. + \tilde{x}_4 + e^{t_2} \prod_{I=1}^4 \tilde{x}_I + e^{t_1-t_2} \prod_{I=1}^4 x_I \right].
\end{aligned} \tag{3.10}$$

One can now see that the fields \tilde{m}_A and \tilde{x}_4 are Lagrange multipliers and their equations of motion set $\tilde{x}_A = -1$ and $x_4 = e^{t_2} - 1$. It is clear from the measure that all the variables except $m_{1,2}$ are “good” variables. The situation can be rectified following a procedure that is often useful for getting a geometric description from the Landau-Ginzburg B-model mirrors of Calabi-Yau manifolds (for instance see [12]). We introduce additional fields (u_a, v_a) , $(a = 1, 2)$ to absorb the non-trivial measure for m_a ⁴. In the resulting expression m_1 and m_2 become Lagrange multipliers enforcing algebraic constraints. Integrating out the Lagrange multipliers we finally arrive at the interesting part of the story

$$Z = e^{t_1} \int \prod_{a=1,2} [dx_a du_a dv_a d\eta_a d\chi_a] \prod_a \delta(u_a v_a + \eta_a \chi_a - x_a + 1) \delta(e^{t_1}(e^{-t_2} - 1)x_1 x_2 - 1). \quad (3.11)$$

The δ -functions inside the integral contain the information on the geometry of the mirror manifold. The first thing to note is that the putative mirror geometry has dimension (3|4) (the six bosonic coordinates have three delta-functions constraints) consistent with the conjecture of [2].

How do we understand and interpret this mirror geometry? One possibility is to take a limit of the A-model Kähler parameters in which a simple description appears. Another possibility might be to homogenize the algebraic constraints and interpret the mirror as a complete intersection in projective superspace⁵. However there is not a unique way to homogenize the equations, and the most obvious possibilities do not result in a Calabi-Yau supermanifold. Therefore we focus on the first possibility and perform a simple rescaling of the fields to rewrite the Landau-Ginzburg “period” as

$$Z = \int \prod_{a=1,2} [du_a dv_a dx_a d\eta_a d\chi_a] \delta(u_1 v_1 + \eta_1 \chi_1 - x_1 + \nu) \delta(u_2 v_2 + \eta_2 \chi_2 - x_2 + \nu) \delta(x_1 x_2 - \mu), \quad (3.12)$$

where $\mu = e^{-t_1}$ and $\nu = (e^{-t_2} - 1)^{1/2}$.

As pointed out earlier, the \mathbf{Z}_2 symmetry under the exchange of t_1 and t_2 , corresponding to the exchange of the two $\mathbb{CP}^{3|3}$ factors in the A-model, has been broken. The Landau-Ginzburg integral (3.12) is a period integral over a supermanifold of dimension (3|4) which

⁴ The idea is to make use of the relation $\int dudve^{uvm} = \frac{1}{m}$ for a suitable choice of contour.

⁵ Even though strictly speaking the constraints describe a non-compact manifold, the original geometry we started from, *i.e.* the quadric, is compact and therefore one expects that what we see in (3.11) is only an affine coordinate patch inside a compact geometry; but one needs to check that the measure is consistent with such an interpretation

we want to identify. (We point out that our identification of the Landau-Ginzburg integral as a “period” is purely a formal analogy with the case of bosonic Calabi-Yaus. For supermanifolds the integrals above vanish unless there are suitable insertions of fermionic coordinates. For a discussion of related issues see [10].) We start by considering the limit $\nu \rightarrow 0 \sim t_2 \rightarrow 0$. Then, solving the constraints for x_1, x_2

$$Z = \int \prod_{a=1,2} du_a dv_a d\eta_a d\chi_a \delta([u_1 v_1 + \eta_1 \chi_1][u_2 v_2 + \eta_2 \chi_2] - \mu), \quad (3.13)$$

we can actually perform the delta-function integral by introducing additional variable changes as follows. In a patch where $u_1 \neq 0$ we can introduce the variables $z_2 = u_2/u_1$, $z_3 = v_1/u_1$, $z_4 = v_2/u_1$, $\psi_1 = \eta_1/u_1$, $\psi_2 = \eta_2/u_1$, $\psi_3 = \chi_1/u_1$ and $\psi_4 = \chi_2/u_1$. With these new variables the Landau-Ginzburg path integral is

$$\begin{aligned} Z &= \int \frac{du_1}{u_1} dz_2 dz_3 dz_4 d\psi_1 d\psi_2 d\psi_3 d\psi_4 \delta(u_1^4 (z_2 - \psi_1 \psi_2)(z_3 z_4 - \psi_3 \psi_4) - \mu) \\ &\equiv \int \frac{du_1}{u_1} \Omega_1 \delta(Au_1^4 - \mu) \\ &= \frac{1}{\mu} \int \Omega_1, \end{aligned} \quad (3.14)$$

where we introduce the form $\Omega_1 = dz_2 dz_3 dz_4 d\psi_1 d\psi_2 d\psi_3 d\psi_4$. Note that various factors of u_1 in the measure, induced by the above variable change, cancel out precisely because of the presence of fermionic coordinates. The form Ω_1 is the natural holomorphic form on $\mathbb{CP}^{3|4}$, given in affine coordinates z_2, z_3, z_4, ψ_I , in a patch where one of the homogeneous coordinates (identified with u_1) is set to 1. We can see that this interpretation is valid by trying to write the integral in a different patch. Then we can solve the δ -function constraints for u_2 , and we will define some new coordinates $\tilde{z}_i, \tilde{\psi}_I$ related to z_i, ψ_I in precisely the way affine coordinates in different patches of $\mathbb{CP}^{3|4}$ are related. This will lead to a holomorphic form Ω_2 in the second affine patch, where Ω_1 and Ω_2 are the same on the intersection of the two patches.

This is a confirmation of the conjecture of [2] that $\mathbb{CP}^{3|4}$ is the mirror supermanifold of the quadric Q and that this interpretation only emerges in a limit of the Kähler moduli of the quadric. The Landau-Ginzburg partition function computed in the mirror manifold, in the limit where $t_2 \rightarrow 0$, is simply proportional to e^{t_1} . This seems to imply that, up to a normalization, the periods (in this limit for t_2) do not depend on the Kähler class t_1 of the original manifold. Whereas this is in contrast to the usual situation in mirror symmetry, it

is consistent with the arguments of [10] , where supermanifolds were proposed as candidates for mirrors of rigid Calabi-Yaus. That could only be possible if in these models the Kähler class decouples from the other observables. This issue deserves further study.

It is also worth pointing out that the discrete symmetry ($t_1 \leftrightarrow t_2$) which exchanges the two $\mathbb{CP}^{3|3}$ s of the quadric is not visible in the mirror geometric description (3.13) obtained from the Landau-Ginzburg dual (3.5). Of course, obtaining the geometric picture required us to integrate out certain fields which then broke the $t_1 \leftrightarrow t_2$ symmetry. Obviously we could follow a different route which would yield the same geometric mirror but with t_1 and t_2 interchanged in Eq.(3.12). It is tempting to speculate that this breaking of the $t_1 \leftrightarrow t_2$ symmetry is intrinsically related to the way these spaces are defined as twistor spaces. In particular, the bosonic part of $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$, in which the quadric is embedded, is the space of all self-dual planes and anti-self-dual planes in \mathbb{C}^4 . On the other hand, the bosonic part of $\mathbb{CP}^{3|4}$ is simply the space of self-dual (or anti-self-dual) planes in \mathbb{C}^4 and thus singles out states of a particular helicity.

Finally, we consider the geometric mirror (3.12) in the general case $\nu \neq 0$. One expects that it should be interpreted as a complex deformation of the twistor superspace. While $\mathbb{CP}^{3|4}$ itself may not have complex deformations, the twistor superspace \mathbb{PT}' which should actually be defined as $\mathbb{CP}^{3|4} \setminus \mathbb{CP}^{1|4}$ [1] , can have complex deformations. This is well-known in the bosonic case [13][14] . It is interesting to note that if we ignore the fermions in the delta-functions in (3.12) and set $\nu = 0$, after a simple variable change it is possible to interpret the resulting expressions as an $\mathcal{O}(1) \oplus \mathcal{O}(1)$ bundle over \mathbb{CP}^1 . Turning on non-zero ν would be a deformation of this bundle. It would be interesting to pursue this interpretation in the presence of the fermions. These issues deserve further attention, particularly if we would like to interpret (3.12) (for generic Kähler parameters of the quadric) as a complex deformation of twistor superspace.

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